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On the Secondary Zeta-Functions

NORMAN LEVINSON*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Using summability it is shown that $\sum_{n\geqslant 2}(\Lambda(n)-1)\, n^{-1/2}(\log n)^{-s}$ defines an entire function in the s-plane. Its asymptotic nature is found and a functional equation relating it to the series $\sum\{i(1/2-\rho)\}^{1-s}$, $\operatorname{Im}\rho=\gamma>0$, is obtained where $\rho=\beta+i\gamma$ are the nontrivial zeros of Riemann's zeta-function.

Guinand [1], assuming the Riemann hypothesis (R.H.) and a further condition showed a relationship in $0<\sigma<1$, $s=\sigma+it$, between

$$\lim_{T\to\infty}\left\{\sum_{n\leqslant T}\Lambda(n)\, n^{-1/2}(\log n)^{-s}-\int_0^T x^{-s}e^{x/2}\,dx\right\} \tag{1}$$

and (with $\rho=\frac{1}{2}+i\gamma$ under R.H.)

$$\lim_{T\to\infty}\left\{\sum_{0<\gamma<T}\gamma^{s-1}-\frac{T^s}{2\pi s}\log\frac{T}{2\pi}+\frac{T^s}{2\pi s^2}\right\}. \tag{2}$$

Chakravarty [2], working with more complicated "infinites" than those subtracted from the series in (1) and (2) and using a summability weighting, showed such a relationship valid without any hypotheses. He called his analog of (1) and (2) the secondary zeta-functions. He later [3] described modifications of these corresponding more closely to Guinand's. Here the two functions Z and Y below will also be called secondary zeta-functions. For $b>0$ let

$$Z(s,b)=\sum_2^\infty(\Lambda(n)-1)\, n^{-1/2}(\log n)^{-s}\exp\{-b\log n\log\log ne\}. \tag{3}$$

THEOREM 1. *For* $\sigma>0$,

$$\lim_{b\to 0} Z(s,b)=Z(s) \tag{4}$$

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is an analytic function and it has an analytic continuation onto the s-plane as an entire function. (The weighting in (3) is Lindelöf summability. Many other types of summability can also be used.)

Let $\rho = \beta + i\gamma$ be, as usual, the zeros of $\zeta(s)$ in $|\sigma| < 1$. Let

$$q = i(\tfrac{1}{2} - \rho) = \gamma + i(\tfrac{1}{2} - \beta)$$

so that under R.H., $q = \gamma$.

THEOREM 2. *For $\sigma > 1$ let*

$$Y(s) = \sum_{\operatorname{Re} q > 0} q^{-s}. \tag{5}$$

Then $Y(s)$ is meromorphic in the s-plane with sole singularities a double pole at $s = 1$ and simple poles at $s = -2n - 1$, $n \geqslant 0$.

The double pole of $Y(s)$ at $s = 1$ is described in [3]. Let

$$L_1(s) = \sum_0^\infty (2n+1)^{-s}, \qquad L_2(s) = \sum_0^\infty (-1)^n (2n+1)^{-s}. \tag{6}$$

Then these are two Dirichlet L-series (mod 4) and $(s-1)L_1(s)$ and $L_2(s)$ are entire functions.

LEMMA 1. *Let $[x]$ represent the integer part of x. For $\sigma < 2$ let*

$$H(s) = \int_0^\infty (e^y - [e^y])\, e^{-y/2} y^{-s}\, dy. \tag{7}$$

Then $H(s)$ is meromorphic in the s-plane with sole singularities simple poles at $s = 2n$, $n \geqslant 1$. Moreover $H(s) = H_1(s) + H_2(s)$ where

$$H_1(s) = \sum_0^\infty (\log 2)^{2n+2-s}/(2n+2-s)\, 2^{2n}(2n+1)! \tag{8}$$

and $H_2(s)$ is entire and

$$\begin{aligned} H_2(s) &= O(2^{-\sigma}\Gamma(1-\sigma)), & \sigma \leqslant 0; \\ &= O(1/(\log 2)^\sigma), & \sigma > 0. \end{aligned} \tag{9}$$

From (8),

$$H_1(s) = O((\log 2)^{-\sigma}/|s-2|), \qquad \sigma < 1$$

THEOREM 3. *The following functional equation is valid in the finite s-plane.*

$$\begin{aligned}\cos \pi s/2\Gamma(s)\, Z(s) \\ = -\pi Y(1-s) - \pi 2^{-s-1}\{L_1(1-s) + L_2(1-s) - 2\}/\sin \pi s/2 \\ + \cos \pi s/2\Gamma(s)\,\{sH(s+1) + \tfrac{1}{2}H(s)\}.\end{aligned} \tag{10}$$

Using $\Gamma(s)\,\Gamma(1-s) = \pi/\sin \pi s$, (10) *takes the form*

$$\begin{aligned}Z(s) = -\,2 \sin \pi s/2\Gamma(1-s)\; Y(1-s) \\ - \Gamma(1-s)\, 2^{-s}\{L_1(1-s) + L_2(1-s) - 2\} + sH(s+1) + \tfrac{1}{2}H(s).\end{aligned} \tag{11}$$

This is the analog of [1, p. 116; 2, (3.29)].

THEOREM 4. *Suppose R.H. is false. Then there exists* $k > 0$ *and* a, $0 < a < k$ *such that there is a finite set of* q *with* $\operatorname{Re} q > 0$, $q_1, q_2, \ldots, q_N$ *such that* $\arg q_j = -k$, *and* $\operatorname{Re} q_j$ *are increasing. Let* $A_1 = -\log(\sin a \log 2)$. *Let* m_j *be the multiplicity of* $\tfrac{1}{2} + iq_j$ *as a zero of* $\zeta(s)$. *Then for* $\sigma \geqslant -\tfrac{1}{2}$, $t \geqslant 0$

$$Z(s) = -2\pi e^{\pi i s/2} \sum_1^N m_j q_j^{1-s}/\Gamma(s) + O\{e^{-(\pi/2-a)|t| + A_1\sigma}\Gamma(\sigma)/\Gamma(s)\}, \tag{12}$$

while for $\sigma \leqslant \tfrac{1}{2}$, $t \geqslant 0$

$$Z(s) = -2\pi e^{\pi i s/2} \sum_1^N m_j q_j^{1-s} \sin \pi s\Gamma(1-s) + O\{e^{(\pi/2+a)|t|}\Gamma(1-s)\}. \tag{13}$$

For $t < 0$, *similar results hold with* $e^{\pi i s/2}$ *replaced by* $e^{-\pi i s/2}$ *and* q_j *by* $\bar{q}_j$. *In particular it is seen that for increasing* $|t|$ *the sums in* (13) *and* (14) *grow as* $e^{k|t|}$, *while the error terms have their exponential growth dominated by* $e^{a|t|}$.

THEOREM 5. *Suppose R.H. is true. Then for* $\sigma > 2$

$$Z(s) = \sum_2^\infty (\Lambda(n) - 1)\, n^{-1/2}(\log n)^{-s} \tag{14}$$

converges. Also for any given $\delta > 0$

$$Z(s) = (\Lambda(2) - 1)\, 2^{-1/2}(\log 2)^{-s} + O(1), \qquad |\arg s| \leqslant \pi/2 - \delta \tag{15}$$

$$Z(s) = (\Lambda(2) - 1)\, 2^{-1/2}(\log 2)^{-s} + O(|s|/\sigma - 2), \qquad \sigma \geqslant 2 + \delta \tag{16}$$

$$Z(s) = O(|t|), \qquad -\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{5}{2}. \tag{17}$$

For $\sigma < -\frac{1}{2}$ since the series for $Y(1-s)$, $L_1(1-s)$, and $L_2(1-s)$ all converge, (11) and Lemma 1 determine the nature of $Z(s)$ (in particular, Y, L_1 and L_2 are all bounded for $\sigma \leqslant -\frac{1}{2}$).

THEOREM 6. *The convergent series for $Y(1-s)$, $\sigma < 0$, gives the nature of $Y(1-s)$ (depending on R.H. or not R.H). For $\sigma > -\frac{1}{2}$ the information about $Z(s)$ in (10) determines $Y(1-s)$.*

According to (10) and (12), under not R.H., the finite sum $\sum m_j(q_j^{1-s} + \bar{q}_j^{1-s})$ determines the size of $Y(1-s)$ for increasing $|t|$ in $\sigma > -\frac{1}{2}$ even though the infinite series representation for $Y(1-s)$ diverges for $\sigma > 0$.

Proof of Theorem 1. Let

$$g(u) = -\zeta'\zeta(\tfrac{1}{2} + u) - \zeta(\tfrac{1}{2} + u) + 1. \tag{18}$$

Then if $w = u + iv$ and with $\sum$ on $\Lambda(n)$ always for $n \geqslant 2$,

$$g(w) = \sum (\Lambda(n) - 1)\, n^{-1/2-w} \qquad u > \tfrac{1}{2}. \tag{19}$$

From (18), $g(u)$ is certainly analytic for real u for $u > -1$. Let

$$G(s) = \int_0^\infty g(u)\, u^{s-1}\, du. \tag{20}$$

Then because of the exponentially falling of $g(u)$, shown by (19), as $u \to \infty$, $G(s)$ is analytic for $\sigma > 0$. Also by Lindelöf summability (see Appendix):

$$g(u) = \lim_{b\to\infty} \sum (\Lambda(n) - 1)\, n^{-1/2-u} \exp\{-b \log n \log\log ne\} \tag{21}$$

uniformly for $0 \leqslant u \leqslant 1$. Hence, if

$$G_{11}(s) = \int_0^1 g(u)\, u^{s-1}\, du, \qquad G_{12}(s) = \int_1^\infty g(u)\, u^{s-1}\, du \tag{22}$$

then $G = G_{11} + G_{12}$. By (19)

$$G_{12}(s) = \sum (\Lambda(n) - 1)\, n^{-1/2} \int_1^\infty n^{-u} u^{s-1}\, du \tag{23}$$

and by the uniformity of (21), for $\sigma > 0$,

$$G_{11}(s) = \lim_{b\to 0} \sum (\Lambda(u) - 1)\, n^{-1/2} \exp\{-b \log n \log\log ne\} \int_0^1 n^{-u} u^{s-1}\, du. \tag{24}$$

Since $n^{-u} \leqslant n^{-u/4}/n^{3/4}$ for $u \geqslant 1$ it follows that for $\sigma \leqslant A$, for any A, the series (23) is uniformly absolutely convergent. Hence,

$$G_{12}(s) = \lim_{b\to 0} \sum (\Lambda(n) - 1)\, n^{-1/2} \exp\{-b \log n \log\log ne\} \int_1^\infty n^{-u} u^{s-1}\, du.$$

Adding this to (24) gives $G(s) = \Gamma(s) \lim_{b\to 0} Z(s, b)$. Thus,

$$Z(s) = G(s)/\Gamma(s) \tag{25}$$

exists as an analytic function in $\sigma > 0$. By (22), $G_{12}(s)$ is an enitre function. Since $g(w)$ is analytic for $|w| < 2$, expanding $g(u)$ in a power series and integrating G_{11} term by term in (22), $\sigma > 0$, gives the analytic continuation of $G_{11}(s)$ as a meromorphic function with simple poles at $s = -n$, $n \leqslant 0$. Hence, $G(s)$ is meromorphic with these poles as its sole singularities. Therefore by (25), $Z(s)$ is an entire function and Theorem 1 is proved.

Proof of Lemma 1. Let

$$H_1(s) = 2 \int_0^{\log 2} \sinh(y/2)\, y^{-s}\, ds, \qquad H_2(s) = \int_{\log 2}^{\infty} (e^y - [e^y])\, e^{-y/2} y^{-s}\, dy.$$

Then $H(s) = H_1(s) + H_2(s)$ is clearly an entire function. Expanding $\sinh y/2$ in a power series and integrating in $H_1(s)$ term by term for $\sigma < 2$ gives (8) and thus the analytic continuation of $H_1(s)$ as a meromorphic function with poles at $s = 2n$, $n \geqslant 1$. The appraisals of (9) are easily demonstrated from

$$|H_2(s)| \leqslant \int_{\log 2}^{\infty} e^{-y/2} y^{-\sigma}\, dy$$

and that of $H_1(s)$ for $\sigma < 1$ from (8).

LEMMA 2. *For* $0 < \sigma < 1$ *let*

$$G_2(s) = \int_0^{\infty} \{-\zeta(\tfrac{1}{2} + u) + 1 + (u - \tfrac{1}{2})^{-1}\}\, u^{s-1}\, du. \tag{26}$$

Then

$$G_2(s) = \Gamma(s)\,(sH(s+1) + \tfrac{1}{2}H(s)) \tag{27}$$

and so $G_2(s)$ *is meromorphic with poles at the real integers only.*

Proof of Lemma 2. As is well known and easily shown

$$\begin{aligned} \zeta(s) - 1 - (s-1)^{-1} &= s \int_1^{\infty} x^{-s-1}([x] - x)\, dx, \qquad \sigma > 0 \\ &= -s \int_0^{\infty} e^{-ys}(e^y - [e^y])\, dy. \end{aligned} \tag{28}$$

Writing the integral over $(0, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$ shows that for large σ

$$\int_0^{\infty} e^{-y\sigma}(e^y - [e^y])\, dy = O(1/\sigma^2). \tag{29}$$

Using (28) in (26),

$$G_2(s) = \int_0^\infty u^{s-1}(u + \tfrac{1}{2})\, du \int_0^\infty e^{-yu-y/2}(e^y - [e^y])\, dy.$$

Because of (29) the inner integral is $O(1/u^2)$ for large u and so the repeated integral is absolutely convergent for $0 < \sigma < 1$. Inverting the order of integration

$$G_2(s) = \int_0^\infty e^{-y/2}(e^y - [e^y])\, dy \int_0^\infty u^{s-1}(u + \tfrac{1}{2})\, e^{-yu}\, du$$

$$= \int_0^\infty e^{-y/2}(e^y - [e^y])\, \{y^{-s-1}\Gamma(s+1) + \tfrac{1}{2}y^{-s}\Gamma(s)\}\, dy$$

which proves (27).

For $0 < \sigma < 1$, let

$$G_1(s) = \int_0^\infty \left(-\frac{\zeta'}{\zeta}(\tfrac{1}{2} + u) - (u - \tfrac{1}{2})^{-1}\right) u^{s-1}\, du. \tag{30}$$

Then by (26),

$$G(s) = G_1(s) + G_2(s). \tag{31}$$

The Mellin transform (30) was treated in [3, (2.9)] and the following lemma is proved there.

LEMMA 3. *For $-1 < \sigma < 1$, $G_1(s)$ is analytic except for a pole at $s = 0$ and for $-1 < \sigma < 0$ is given by*

$$G_1(s) = -2^{-s}\pi(L_1(1-s) + L_2(1-s) - 2)/\sin \pi s - \pi Y(1-s)/\cos(\pi s/2). \tag{32}$$

Proof of Lemma 3. Integrate (30) by parts to get

$$sG_1(s) = \int_0^\infty \left\{\frac{d}{du}\frac{\zeta'}{\zeta}(\tfrac{1}{2} + u) - (u - \tfrac{1}{2})^{-2}\right\} u^s\, du. \tag{33}$$

The integral now represents an analytic function for $-1 < \sigma < 1$. As is well known,

$$\frac{d}{ds}\frac{\zeta'}{\zeta}(s) = (s-1)^{-2} - \sum (s-\rho)^{-2} - \sum_0^\infty (s + 2n)^{-2}.$$

Hence (33) becomes

$$sG_1(s) = -\int_0^\infty \left\{2 \sum_{\mathrm{Re}\, q > 0} \frac{u^2 - q^2}{(u^2 + q^2)^2} + \sum_0^\infty (u + 2n + \tfrac{5}{2})^{-2}\right\} u^s\, du.$$

This can be integrated termwise for $-1 < \sigma < 0$ since the series converges uniformly and is $O(\log u/u)$ for large u. For $\operatorname{Re} a > 0$ and $-1 < \sigma < 1$

$$\int_0^\infty u^s(u+a)^{-2}\,du = \pi s a^{s-1}/\sin \pi s$$

$$\int_0^\infty u^s(u^2-a^2)\,(u^2+a^2)^{-2}\,du = \pi s a^{s-1}/(2\cos \pi s/2).$$

Thus, for $-1 < \sigma < 0$

$$G_1(s) = -\pi Y(1-s)/\cos(\pi s/2) - \pi 2^{1-s} \sum_0^\infty (4n+5)^{s-1}/\sin \pi s$$

and this proves the lemma since $sG_1(s)$ is analytic for $-1 < \sigma < 1$ and the analytic continuation of $Y(1-s)$ as a meromorphic function is provided on $-1 < \sigma < 1$.

Proof of Theorem 3. Since $G(s)$ and $G_2(s)$ are meromorphic in the finite s-plane it follows from (31) that $G_1(s)$ must be. From (25), (31), (27), and (32), Theorem 3 follows.

Proof of Theorem 2. From (10) and the simple pole of $L_1(1-s)$ at $s=0$, it follows that $Y(1-s)$ has a double pole there. The simple poles of $Y(1-s)$ at $s=2n$ follow from the poles of $H(s)$ at $s=2n$ and the zeros of $\sin \pi s/2$ in the denominator. (The poles of $H(s+1)$ are cancelled by the zeros of $\cos \pi s/2$.) The series definition (5) shows $Y(1-s)$ is analytic for $\sigma < 0$.

Proof of Theorem 4. Since R.H. is assumed false and $\rho = \frac{1}{2} + iq$, there exists $k > 0$ such that $\min \arg q = -k$. There can be only a finite number of q, $\operatorname{Re} q > 0$, for which this is true since $|\operatorname{Im} q| < \frac{1}{2}$. Denote these q in increasing size of $\operatorname{Re} q$ by $q_1, q_2, \ldots, q_N$. Let the multiplicity of q_j be $m_j \geqslant 1$. Choose a so that $0 < a < k$ and so that all q, $\operatorname{Re} q > 0$, not in the above set $q_1, \ldots, q_N$ have $\arg q > -a$ so that there is no q with $-k < \arg q \leqslant -a$. For $\sigma > 0$ rotate the line of integration in (20) through an angle of $\pi/2 - a$ which can be justified by (19). Then with $w = r\exp\{i(\pi/2 - a)\}$, from (18),

$$G(s) = \exp\{i(\pi/2-a)\,s\} \int_0^\infty g(rie^{-ia})\, r^{s-1}\,dr + S_1(s) \tag{34}$$

where, since $\rho_j - \frac{1}{2} = iq_j$,

$$S_1(s) = -2\pi e^{\pi is/2} \sum_1^N m_j q_j^{s-1}. \tag{35}$$

Let

$$\int_0^\infty g(rie^{-ia})\, r^{s-1}\,dr = I_1(s) + I_2(s), \tag{36}$$

where

$$I_1(s) = \int_0^\infty g(rie^{-ia})\, r^{s-1}\, dr, \qquad I_2(s) = \int_1^\infty g(rie^{-ia})\, r^{s-1}\, dr. \tag{37}$$

Since $g(w)$ is analytic for $|w| < \frac{5}{2}$ it follows that

$$g(w) = \sum_0^\infty a_n w^n, \qquad a_n = O(2^{-n}). \tag{38}$$

Thus,

$$I_1(s) = \sum_0^\infty a_n i^n e^{-ian}/(s+n) \tag{39}$$

and so it is meromorphic in the s-plane with poles at $s = -n$, $n \geqslant 0$. Since by (19), $I_2(s)$ is entire, it now follows from (34) that for all s

$$G(s) = \exp\{i(\pi/2 - a)\, s\}\, (I_1(s) + I_2(s)) + S_1(s). \tag{40}$$

From (37) and (19) with $A_1 = -\log(\sin a \log 2)$

$$\begin{aligned} I_2(s) &= O(e^{A_1\sigma}\Gamma(\sigma)) && \sigma \geqslant \tfrac{1}{2}, \\ &= O(1) && \sigma \leqslant \tfrac{1}{2}. \end{aligned} \tag{41}$$

Thus, for $t \geqslant 0$ from (38), (39), (40), and (41) and $Z = G/\Gamma$

$$\begin{aligned} Z(s) = -2\pi e^{\pi is/2} \sum m_j q_j^{s-1}/\Gamma(s) &+ O(e^{-(\pi/2-a)|t|}) \\ &\times \left(I_2(s)/\Gamma(s) + \sum_0^\infty 2^{-n}/|s+n|\,\Gamma(s)\right) \end{aligned}$$

so for $\sigma \geqslant -\frac{1}{2}$, $t \geqslant 0$

$$Z(s) = -2\pi e^{\pi is/2} \sum m_j q_j^{s-1}/\Gamma(s) + O(1)\, e^{-(\pi/2-a)|t|+A_1\sigma}\Gamma(\sigma)\,\Gamma(s) \tag{42}$$

and for $\sigma \leqslant \frac{1}{2}$, $t \geqslant 0$, with $\Gamma(s) = \pi/\sin \pi s \Gamma(1-s)$

$$Z(s) = -2e^{\pi is/2}\Gamma(1-s) \sin \pi s \sum m_j q_j^{s-1} + O(e^{(\pi/2+a)|t|})\,\Gamma(1-s). \tag{43}$$

For $t < 0$ the only change is $e^{\pi is/2}$ goes to $e^{-\pi is/2}$ and q_j to $\bar{q}_j$. In particular, as $t \to \infty$, the first term is exponentially larger than the error term for increasing $|t|$. For large positive σ, q_N^{s-1} dominates while for σ very negative q_1^{s-1} dominates and both are as large as e^{kt} while the last term has in its place $O(e^{at})$, $a < k$.

Proof of Theorem 5. R.H. holds. Proceeding with $\Sigma\Lambda(n)\, n^{-1/2}$, $n < x$, much as is done in Landau [4, p. 388] or Inghan [5, p. 83] for $\Sigma\Lambda(n)$, $n \leqslant x$, yields

$$F(x) = \sum_{n \leqslant x} (\Lambda(n) - 1)\, n^{-1/2} = O(\log^2 x).$$

Hence, for $\sigma > 2$

$$\begin{aligned}\sum_{n \geqslant 5} (\Lambda(n) - 1)\, n^{-1/2}(\log n)^{-s} &= \sum_{n \geqslant 5} F(n)\, ((\log n)^{-s} - (\log(n+1))^{-s}) \\ &\quad - F(4)\, (\log 4)^{-s} \\ &= s \sum_{n \geqslant 5} F(n) \int_n^{n+1} (\log v)^{-s-1}\, dv/v + O(e^{-\sigma/4}) \\ &= s \sum_{n \geqslant 5} \frac{O(\log^2 n)}{n(\log n)^{\sigma+1}} + O(e^{-\sigma/4}).\end{aligned}$$

Thus,

$$Z(s) = (\log 2 - 1)\, 2^{-1/2}(\log 2)^{-s} + O(s/(\sigma - 2)).$$

This proves (15) and (16).

By Lemma 1 on H, for $\sigma < 0$, (11) gives information on $Z(s)$ since the series for Y, L_1, and L_2 all converge. There is a gap for $0 \leqslant \sigma \leqslant 2$. This is, however, easily filled in by a Phragmen–Lndelöf theorem. Indeed for $\sigma = -\frac{1}{2}$, (11) shows

$$Z(-\tfrac{1}{2} + it) = O(|\, t\, |) \tag{44}$$

and by (16)

$$Z(\tfrac{5}{2} + it) = O(|\, t\, |). \tag{45}$$

Now the results (42) and (43) are valid under R.H. if the series $\sum m_j q_j^{s-1}$ is deleted. Thus,

$$Z(\sigma + it) = O(e^{a|t|}), \qquad -\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{5}{2}.$$

But this and (44) and (45) proves (17).

Appendix

The summability used in (21) corresponds to Lindelöf summability [6, p. 197] in the case of power series. Since I have no reference for its use on Dirichlet series (or equivalently, Laplace transforms), a brief treatment will now be given. Lindelöf's work was done in 1903. The summability methods of LeRoy or Mittag–Leffler [6, p. 197] could also be used.

THEOREM A. *Let*

$$F(s) = \sum a_n/n^s, \qquad \sum |a_n| n^{3/2} < \infty. \tag{46}$$

Let $F(s)$ have an analytic continuation to $w = s_0 = \sigma_0 + it_0$ along the line segment $t = t_0$, $\sigma_0 \leqslant \sigma < 2$. Then uniformly for $\sigma_0 \leqslant \sigma \leqslant 2$, $t = t_0$,

$$F(s) = \lim_{b \to 0} \sum a_n n^{-s} \exp(-b \log n \log \log ne)$$

where $b > 0$ above.

(The use of $\frac{3}{2}$ in (46) has no special significance.)

LEMMA A1. *Let $\sigma < 3$. Let*

$$J = \sum a_n n^{-s} \exp(-b \log n \log \log ne).$$

Then

$$J = \int_0^\infty \frac{d^2}{dx^2} [\exp(-bx \log(1+x))]\, dx \frac{1}{2\pi i} \int_{-i\infty+3}^{i\infty+3} \frac{F(w) e^{x(w-s)}}{(w-s)^2}\, dw.$$

Proof of Lemma A1. Let

$$I_n = \int_0^\infty \frac{d^2}{dx^2} [\exp(-bx \log(1+x))]\, dx \frac{1}{2\pi i} \int_{-i\infty+3}^{i\infty+3} \frac{n^{-w} e^{x(w-s)}}{(w-s)^2}\, dw.$$

By residue theory, the inner integral is 0 for $x \leqslant \log n$ and is $n^{-s}(x - \log n)$ for $x \geqslant \log n$. Hence,

$$I_n = n^{-s} \int_{\log n}^\infty \frac{d^2}{dx^2} [\exp(-bx \log(1+x))]\, (x - \log n)\, dx.$$

Integrate by parts twice to get

$$I_n = n^{-s} \exp(-b \log n \log \log ne).$$

Multiply by a_n and add to prove the lemma.

LEMMA A2. *Let a and c be real. Let k_1 be 0, 1 or 2. Let k_2 and k_3 be real and nonnegative. Let*

$$I = \int_0^\infty \exp[-bx \log(1+x) + (a+ic)x] \frac{\log^{k_1}(1+x)\, x^{k_2}}{(1+x)^{k_3}}\, dx.$$

If $|c| > 0$, then for $b < |c|/10$

$$|I| \leqslant \Gamma(k_1 + k_2 + 1)\,(2/|c|)^{k_2+2k_1+1} + 4\Gamma(k_2+1)\,(2/|c|)^{k_2+1}.$$

Proof of Lemma A2. By taking the complex conjugate it suffices to take $c > 0$. Replace x by z and deform the path of integration onto $z = iy$ to get

$$|I| \leqslant \int_0^\infty \exp(by\pi/2 - cy)\{\log^{k_1}(1 + y^2)^{1/2} + (\pi/2)^{k_1}\} y^{k_2}\, dy.$$

Since $b < c/10$

$$|I| \leqslant \int_0^\infty e^{-cy/2}(y^{2k_1+k_2} + 4y^{k_2})\, dy$$

which proves the lemma.

Proof of Theorem A. By hypothesis $F(w)$ is analytic in an open set containing the line segment $v = t_0$, $\sigma \leqslant u \leqslant 3$ and so for some $\delta > 0$, $F(w)$ is analytic in the rectangle $|v - t_0| \leqslant \delta$, $\sigma_0 - \delta < v < \infty$. Let C denote the contour consisting of the half-lines $u = 3$, $-\infty < v < t_0 - \delta$ and $t_0 + \delta < v < \infty$; the segments $v = t_0 \pm \delta$, $\sigma_0 - \delta \leqslant u \leqslant 3$; and the segment $u = \sigma_0 - \delta$, $t_0 - \delta \leqslant v \leqslant t_0 + \delta$. Taking account of the residue at $w = s$,

$$\frac{1}{2\pi i}\int_{-i\infty+3}^{i\infty+3} \frac{F(w)\, e^{(w-s)x}}{(w-s)^2}\, ds = xF(s) + F'(s) + \frac{1}{2\pi i}\int_C \frac{F(w)\, e^{x(w-s)}}{(w-s)^2}\, dw.$$

By Lemma A1

$$J = J_1 + J_2,$$

where

$$J_1 = \int_0^\infty \frac{d^2}{dx^2}\exp(-bx\log(1+x))\,(xF(s) + F'(s))\, dx$$

and (computing the second derivative)

$$J_2 = \frac{b}{2\pi i}\int_C \frac{F(w)}{(w-s)^2}\, dw \int_0^\infty \exp\{-bx\log(1+x) + x(w-s)\}$$

$$\times \left[b\log^2(1+x) + \frac{2bx\log(1+x)}{1+x} + \frac{bx^2}{(1+x)^2} - \frac{2+x}{(1+x)^2}\right] dx.$$

Integrating by parts,

$$J_1 = F(s).$$

C is divided into three parts, C_1, C_2, and C_3. C_2 is the line segment $u = \sigma_0 - \delta$, $|v - t_0| < \delta$; C_1 is the part of C for which $v \geqslant t + \delta$; and C_3 the part for which $v \leqslant t - \delta$.

Now let

$$J_2 = J_{21} + J_{22} + J_{23},$$

where

$$J_{2n} = \frac{b}{2\pi i}\int_{C_n}\frac{F(w)}{(w-s)^2}\,dw\int_0^\infty \exp\{-bx\log(1+x) + x(w-s)\}$$
$$\times\left[b\log^2(1+x) + \frac{2bx\log(1+x)}{1+x} + \frac{bx^2}{(1+x)^2} - \frac{2+x}{(1+x)^2}\right]dx.$$

Since in C_2, $\mathrm{Re}(w-s) \leqslant -\delta$ and since $\log(1+x) \leqslant x$

$$|J_{22}| \leqslant \frac{b}{2\pi}\int_{C_2}\frac{|F(w)|}{\delta^2}\,|dw|\int_0^\infty e^{-\delta x}(4bx^2+2)\,dx.$$

As $b \to 0$, $J_{22} \to 0$. Since on C_1, $\mathrm{Im}(w-s) = v - t_0 \geqslant \delta$, it follows from Lemma A2 that the inner integral of J_{21} is bounded as $b \to 0$. Hence, $J_{21} \to 0$ as $b \to 0$. A similar result holds for J_{23} and Theorem A is proved.

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